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Precise bounds for the sequential order of products of some Fréchet topologies

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Abstract

The sequential order of a topological space is the least ordinal for which the corresponding iteration of the sequential closure is idempotent. Lower estimates for the sequential order of the product of two regular Fréchet topologies and upper estimates for the sequential order of the product of two subtransverse topologies are given in terms of their fascicularity and sagittality. It is shown that for every countable ordinal α, there exists a Lašnev topology such that the sequential order of its square is equal to α. © 1998 Elsevier Science B.V.

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Introduction

The sequential order $\sigma(x)$ at a point x of a topological space X is the least ordinal α such that whenever x belongs to an iterated sequential closure of a set, then it belongs to its α-iterated sequential closure. The sequential order of X is equal to $\sup_{x\in X} \sigma(x)$.

Sequential order is always less than or equal to ω_1. Recall that a topology is sequential if each sequentially closed set is closed. Fréchet topologies are precisely the sequential topologies of sequential order less than or equal to 1. It is well known [1,9,7,12] that the product of two Fréchet topologies needs neither be sequential nor of order less than or equal to 1. This paper is devoted to the study of the sequential order of products of Fréchet topologies.

In [13] Nogura and Shibakov investigate the sequential order of products of sequential topologies under the requirement that the products be also sequential. They prove in

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particular that if the product of two Fréchet topologies admitting point countable k-networks [1] is sequential, then its sequential order is less than or equal to 2. On the other hand they construct [13, Example 2.13] two Fréchet topologies with pointwise countable k-networks such that the sequential order (in our sense) of their product equals 3.

Here we show that for every countable ordinal α, there exists a Lašnev space [2] whose square is of sequential order α (as Lašnev spaces are Fréchet spaces with point countable k-networks, our square is not sequential for $\alpha > 2$, because of the above-mentioned result).

In [14] Nogura and Shibakov construct under CH, for each $\alpha \leqslant \omega_1$, two strongly Fréchet topologies the product of which is sequential and of sequential order α.

We express the sequential order of product topologies in terms of fascicularity and sagittality of the component topologies. With every point x of a topological space we associate fascicularity $\lambda(x)$ and sagittality $\mu(x)$. The first corresponds to the rank of multifans converging to x, the second to the rank of arrows (i.e., sequences of multifans) converging to x. If X, Y are regular Fréchet topological spaces, then the sequential order $\sigma(x, y)$ is not less than

$$1 + \left[\left(\lambda(x) \wedge \mu(y)\right) \vee \left(\mu(x) \wedge \lambda(y)\right)\right]$$

for every $x \in X$ and $y \in Y$. The above quantity is an upper bound for the sequential order $\sigma(x, y)$ provided that X and Y are sequential and subtransverse (we say that a topological space X is subtransverse if for every injective sequence (x_n) converging to x, there exists a subsequence (n_k) and a sequence Q_k with $Q_k \in \mathcal{N}(x_{n_k})$ such that for each neighborhood Q of x, there is k_Q for which $Q_k \subset Q$ as $k \geqslant k_Q$). Lašnev spaces are Fréchet subtransverse and normal, so that in case of Lašnev spaces the above quantity is equal to the sequential order $\sigma(x, y)$.

Our method hinges on the following general characterization: if α is the least ordinal such that x belongs to the α-iteration of the sequential closure of a set A, then there exists a multisequence of rank α on A which converges to x.

All the topologies considered throughout this paper are Hausdorff.

1. Sequential order and admissible multisequences

We denote by $\mathrm{cl}_{\mathrm{seq}} A$ the *sequential closure* of A, i.e., the union of the limits of all convergent sequences valued in A. One defines $\mathrm{cl}^0_{\mathrm{seq}} A = A$ and for each ordinal $\alpha > 0$,

$$\mathrm{cl}^\alpha_{\mathrm{seq}} A = \mathrm{cl}_{\mathrm{seq}} \bigcup_{\beta<\alpha} \mathrm{cl}^\beta_{\mathrm{seq}} A.$$ [3]

[1] A family $\mathcal{A}$ of sets, closed under finite unions is a k-network if for each compact set K and each open O with $K \subset O$, there is $A \in \mathcal{A}$ such that $K \subset A \subset O$.

[2] I.e., a closed image of metrizable space.

[3] Some authors, e.g., Nogura and Shibakov [13], define limit powers by $\mathrm{cl}^\alpha_{\mathrm{seq}} A = \bigcup_{\beta<\alpha} \mathrm{cl}^\beta_{\mathrm{seq}} A$.

The least ordinal α for which $\mathrm{cl}_{\mathrm{seq}}^{\alpha}$ is idempotent is called the *sequential order* of the topological space and is denoted by $\sigma(X)$.

If $x \in \mathrm{cl}_{\mathrm{seq}}^{\omega_1} A$, then the *sequential order* $\sigma(x; A)$ (of x with respect to A) is the least ordinal α such that $x \in \mathrm{cl}_{\mathrm{seq}}^{\alpha} A$. Remark that for every x and A, one has $\sigma(x; A) < \omega_1$. The sequential order $\sigma(x)$ is defined by $\sigma(x) = \sup\{\sigma(x; A)\colon\ A \subset X,\ x \in \mathrm{cl}_{\mathrm{seq}}^{\omega_1} A\}$. Consequently, $\sigma(X) = \sup_{x \in X} \sigma(x)$.

Proposition 1.1. *If $x \in \mathrm{cl}_{\mathrm{seq}}^{\omega_1} A$, then*

$$\sigma(x; A) = \min_{\mathrm{cl}_{\mathrm{seq}}^{\omega_1} A \ni x_n \to x} \liminf_n \bigl(\sigma(x_n; A) + 1\bigr). \tag{1.1}$$

Proof. Let (x_n) be a sequence on $\mathrm{cl}_{\mathrm{seq}}^{\omega_1} A$ converging to x. Denote $\alpha_n = \sigma(x_n; A)$ and $\alpha = \liminf_n (\alpha_n + 1)$. Then there exists a subsequence (x_{n_k}) such that $\alpha = \sup_k (\alpha_{n_k} + 1)$, hence

$$x \in \mathrm{cl}_{\mathrm{seq}} \bigcup_k \mathrm{cl}_{\mathrm{seq}}^{\alpha_{n_k}} A \subset \mathrm{cl}_{\mathrm{seq}} \bigcup_{\beta < \alpha} \mathrm{cl}_{\mathrm{seq}}^{\beta} A = \mathrm{cl}_{\mathrm{seq}}^{\alpha} A,$$

so that $\sigma(x; A) \leqslant \alpha$. On the other hand, if $\sigma(x; A) = \alpha$, then by definition there exists a sequence (x_n) converging to x and such that $\alpha = \lim_n (\sigma(x_n; A) + 1)$ with $\sigma(x_n; A) < \sigma(x; A)$. $\square$

Consider the set $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ of finite sequences valued in $\mathbb{N}$ ordered by inclusion (denoted by $\sqsubseteq$). In what follows, (t, s) denotes the concatenation of the finite sequences t and s. It follows that $r \sqsubseteq s$ whenever there exists t such that $s = (r, t)$.

Following Fremlin [8] we consider the subsets T of $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ that are *well-capped* trees (i.e., such that every nonempty subset of T has a maximal element in T)[4] that fulfill

$$s \sqsubseteq t,\ t \in T \Longrightarrow s \in T, \tag{1.2}$$

$$\forall_{t \in T} \Bigl(\exists_{n \in \mathbb{N}} (t, n) \in T \Longrightarrow \forall_{n \in \mathbb{N}} (t, n) \in T \Bigr). \tag{1.3}$$

Denote by $l(t)$ the length of the finite sequence t. Every well-capped tree T admits the unique rank function:

$$r(t) = r(t; T) = \min\Bigl\{\alpha \in \mathrm{Ord}\colon\ \forall_{s \sqsupset t} r(s) < \alpha\Bigr\}.$$

If T is a well-capped tree in $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ fulfilling (1.2) and (1.3), then

$$t \in \max T \Longrightarrow r(t) = 0, \qquad t \notin \max T \Longrightarrow r(t) = \sup_{n \in \mathbb{N}} \bigl(r(t, n) + 1\bigr). \tag{1.4}$$

Of course, if a tree T is *monotone*,[5] i.e., has the property that for every $t \notin \max T$, the sequence $r(t, n)$ is increasing, then

$$r(t) = \lim_n \bigl(r(t, n) + 1\bigr). \tag{1.5}$$

[4] In other words, T considered with the inverse order is well-founded.

[5] Each tree includes a monotone tree of the same rank.

From now on we understand by a *tree* a well-capped tree in $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ fulfilling (1.2), (1.3). The elements of a tree are called *indices*. By a *subtree* we understand a subset of a tree which is a tree in the above sense.

For each $\alpha < \omega_1$, there exists a tree T of rank α (i.e., $r(\emptyset; T) = \alpha$) [8].

Let T be a tree. We define on T the *irreducible convergence*: $\lim_k t_k = t$ if and only if for almost all k, either $t_k = t$ or $t_k = (t, n_k)$ with $\lim_k n_k = \infty$. This convergence is Urysohn.[6] The associated topology,[7] i.e., the finest topology coarser than the irreducible convergence is called the *irreducible topology*.

An Urysohn convergence on T is said to be *admissible* if it is coarser than the irreducible convergence and if for every $t \in T \setminus \max T$, one has $\lim_k t_k = t$ implies that $t_k \sqsupseteq t$ and if moreover (t_k) is such that $t_k \sqsupseteq (t, n_k)$, then $\lim_k n_k = \infty$ and

$$\liminf_k \bigl(r(t_k) + 1\bigr) = r(t). \tag{1.6}$$

The topology associated with an admissible convergence is called *admissible*. Of course, if $r(\emptyset) < \omega_0$, then the only admissible convergence is that irreducible.

Because of (1.6) and (1.4), for every t in a monotone tree T equipped with the irreducible topology,

$$\sigma(t; \max T) = r(t; T). \tag{1.7}$$

A *multisequence* in X is a mapping $f : \max T \to X$, where T is a tree. An *extended multisequence* in X is a map from a tree T to X; the *extension* of f is $\tilde{f} : T \to X$ such that $\tilde{f}(t) = f(t)$ for every $t \in \max T$. We shall use the term *multisequence* also for extended multisequences if no confusion is probable.[8] The *rank* $r(f)$ of a multisequence f is, by definition, the rank of the underlying tree. The *initial restriction* of a multisequence $f : T \to X$ is the restriction of f to a subtree S of T.

An (extended) multisequence $g : S \to X$ is a *transmultisequence* of $f : T \to X$ if there exists a mapping $h : S \to T$ such that $g = f \circ h$ and

$$h(\emptyset) = \emptyset, \tag{1.8}$$

$$\forall_{s\in S}\ h(s,n) \sqsupseteq \bigl(h(s), m_n\bigr) \quad \text{with } \lim_n m_n = \infty, \tag{1.9}$$

$$h(\max S) \subset \max T. \tag{1.10}$$

A *submultisequence* of $f : T \to X$ is a transmultisequence such that

$$\forall_{s\in S}\ h(s,n) = \bigl(h(s), m_n\bigr) \quad \text{with } \lim_n m_n = \infty. \tag{1.11}$$

An (extended) multisequence $f : T \to X$, valued in a topological space X, *converges* to a point x if for every $t \in T \setminus \max T$, $\lim_n f(t,n) = f(t)$ and $x = f(\emptyset)$. The *sequential*

[6] A sequence convergence is *Urysohn* if $\lim_n x_n = x$ and $\lim_k n_k = \infty$ imply $\lim_k x_{n_k} = x$ and, if a sequence does not converge to x, then there exists a subsequence such that none of its subsequences converges to x.

[7] The associated topologies of Urysohn convergence with the unicity of limits are sequential [10].

[8] We are grateful to Professor A. Kato for having drawn our attention to [2,11] were (extended) multisequences with some extra topological properties were introduced.

order of a convergent multisequence f is defined by $\sigma(f) = \sigma(f(\emptyset); f(\max T))$. The sequential order $\sigma(f)$ is always less than or equal to the rank $r(f)$.

An injective convergent multisequence $f : T \to X$ is said to be *irreducible* (respectively *admissible*) if the initial convergence on T with respect to f is irreducible (respectively admissible).

One might suspect that if $\sigma(x, A) = \alpha$, then there exists an irreducible multisequence $f : \max T \to A$ that converges to x and such that $r(f) = \alpha$. This is in general not the case.

Example 1.2. *A point of sequential order ω_0 with no irreducible multisequence of rank > 1 converging to it.* Let T be a tree of rank ω_0. Define $\lim_k t_k = \emptyset$ whenever for almost each k, either $t_k = \emptyset$ or $t_k \sqsupseteq n_k$, $\lim_k n_k = \infty$ and there exists $m \in \mathbb{N}$ such that $l(t_k) \leqslant m$; if $t \neq \emptyset$, then $\lim_k t_k = t$ whenever for almost each k, either $t_k = t$ or $t_k = (t, n_k)$, $\lim_k n_k = \infty$. This is an admissible convergence.

A set P is an open neighborhood of $\emptyset$ if and only if there is $h : \mathbb{N} \to \mathbb{N}$ such that $\lim_n h(n) = \infty$ and $\{t\colon\ t \sqsupseteq n \Rightarrow l(t) \leqslant h(n)\} \subset P$ and for each $t \in P \setminus \max T$, there is $n(t)$ with $\{(t, n)\colon\ n \geqslant n(t)\} \subset P$.

Since the topology is coarser than the irreducible topology of T, the sequential order $\sigma(\emptyset; \max T) \leqslant \omega_0$. On the other hand, the sequential order is not finite: if $f : S \to T$ were a monotone multisequence of rank $m < \omega_0$ with $f(\max S) \subset \max T$ and converging to $\emptyset$, then there would exist $k < \omega_0$ such that $l(f(n)) \leqslant k$; hence for each $s \in S$ and $s \sqsupseteq n$, one has

$$l(f(s)) \leqslant l(f(n)) + r(n; S) \leqslant k + m - 1 < \omega_0.$$

There is no irreducible multisequence converging to $\emptyset$ and of rank greater than 1. Suppose that $f : S \to T$ is a monotone multisequence with $\lim_n f(n) = \emptyset$. Then there is $m < \omega_0$ such that $\sup_n l(f(n)) \leqslant m$, consequently for each $k \in \mathbb{N}$, $l(f(n, k)) \leqslant m + 1$, so that $\lim_n f(n, n) = \emptyset$ contradicting irreducibility.

Theorem 1.3. *If $\sigma(x, A) = \alpha$, then there exists a monotone admissible multisequence $f : \max T \to A$ that converges to x and such that $r(f) = \alpha$.*

Proof. If $\sigma(x; A) = 0$, then $x \in A$ and thus $f(\emptyset) = x$ constitutes a multisequence of rank 0. If $\sigma(x; A) = 1$, then $x \in \mathrm{cl}_{\mathrm{seq}} A \setminus A$; hence there is a sequence $(f(n))$ on A converging to x and such that $f(n) \neq f(m)$ for $n \neq m$. Let $\alpha > 1$ and assume that the theorem holds for all $\beta < \alpha$ and $\sigma(x; A) = \alpha$. Then there exists an injective sequence (x_n) that converges to x and such that $(\sigma(x_n; A))$ is monotone and $\lim_n(\sigma(x_n; A) + 1) = \alpha$. Because X is a Hausdorff space, there is a sequence $(P_n)_n$ of mutually disjoint neighborhoods of (x_n), respectively and, by the inductive assumption, for each n, there exists a monotone admissible multisequence f_n on $T_n = \{t\colon\ (n, t) \in T\}$ that converges to x_n such that $r(f_n) = \sigma(x_n; A)$ and $f_n(\max T_n) \subset A \cap P_n$.

We define the extended multisequence

$$f : \{\emptyset\} \cup \bigcup_n \{(n, t)\colon\ t \in T_n\} \to X$$

by putting $f(\emptyset) = x$, $f(n) = x_n$ and $f(n,t) = f_n(t)$ for each $t \in T_n$. This multisequence is injective and, of course, $r(f) = \alpha$. By construction, the sequence convergence on f is coarser than the irreducible convergence. Moreover, each sequence converging to $f(n)$ is eventually in P_n, so that to see that it is an admissible convergence, it is enough to consider the case where $f(t_k)$ converges to $f(\emptyset)$, $t_k \sqsupset n_k$ with $\lim_k n_k = \infty$. Then $\sigma(f(t_k); A) = r(t_k) \leqslant r(n_k)$ and thus by (1.1) and (1.5),

$$\sigma(f(\emptyset); A) \leqslant \liminf_k \bigl(r(t_k) + 1\bigr) \leqslant \liminf_k \bigl(r(n_k) + 1\bigr) = r(t) = \sigma\bigl(f(\emptyset); A\bigr),$$

thus (1.6) holds so that the convergence is admissible.

Corollary 1.4. *If $\sigma(x; A) < \omega_0$, then there exists an irreducible multisequence f on A converging to x and such that $\sigma(f) = r(f) = \sigma(x; A)$.*

2. Multifans and arrows

A convergent multisequence $f : T \to X$ is called a *multifan* if for each t of even length in $T \setminus \max T$, one has $f(t,n) = f(t)$ for each $n \in \mathbb{N}$. A convergent multisequence $f : T \to X$ is said to be an *arrow* if for every t in $T \setminus \max T$ of odd length, one has $f(t,n) = f(t)$ for each $n \in \mathbb{N}$. In other words, f is an arrow if for each n, the restriction of f to $T_n := \{s\colon\ (n,s) \in T\}$ is a multifan. A multifan (respectively arrow) $f : T \to X$ is *injective* if it is injective modulo the equivalence relation: if t is of even (respectively odd) length in $T \setminus \max T$, then $t \equiv (t,m)$ for every $m \in \mathbb{N}$. Let $f : T \to X$ be a multifan and R the subtree of T obtained by removing all maximal indices of odd length. If $f : R \to X$ is injective, then we define its *fascicularity* $\lambda(f)$ as the rank $r(\emptyset; R)$.

Similarly, if $f : T \to X$ is an arrow and if f restricted to the subtree R of T obtained by removing all maximal indices of even length is injective, then we define its *sagittality* $\mu(f)$ as the rank $r(\emptyset; R)$. If $R = \emptyset$, then we convene that $\mu(f) = -1$.

Consequently, if f is a multifan and if g is an arrow, then

$$\lambda(f) \leqslant r(f) \leqslant 1 + \lambda(f), \qquad \mu(g) \leqslant r(g) \leqslant 1 + \mu(g). \tag{2.1}$$

If f is a monotone multifan (i.e., if the corresponding tree is monotone) and f_n is its nth arrow, then $\lambda(f) = \lim_n(\mu(f_n) + 1)$; if g is a monotone arrow and g_n is its nth multifan, then $\mu(f) = \lim_n(\lambda(f_n) + 1)$.

An injective multifan $f : T \to X$ is *untraversable* if for every t of even length, and each $t_k \sqsupset (t, n_k)$ such that $\lim_k f(t_k) = f(t)$, the sequence (n_k) is bounded. In particular, a fan is untraversable if no sequence $(x_{(n_p,k_p)})_p$ with n_p tending to ∞ converges to x. Untraversable fans are frequently denoted by S_ω.

The *fascicularity* $\lambda(x)$ of a point x is the least upper bound of $\lambda(f)$ of all the untraversable multifans f converging to x. The *sagittality* $\mu(x)$ is the least upper bound of $\mu(g)$ of all the untraversable arrows g converging to x. These bounds do not change if we consider only the monotone untraversable multifans and arrows. Therefore and because every untraversable multifan is composed of untraversable arrows, $\mu(x) + 1 \geqslant \lambda(x)$.

3. Lower bounds for regular topologies

We say that a multisequence $f:T \to X$ is *transversally closed* if for each $t \in T \setminus \max T$, $t_k \sqsupseteq (t, n_k)$ such that $\lim_k f(t_k) = x$ and $\lim_k n_k = \infty$ implies that $x = f(t)$.

It follows from [15, Theorem 3.8] of Nogura and Tanaka that for each untraversable fan $(x_{(n,k)})$ converging to x in a regular Fréchet space, there exists a mapping $h:\mathbb{N} \to \mathbb{N}$ such that $\{x\} \cup \{x_{(n,k)}\colon k \geqslant h(n),\ n \in \mathbb{N}\}$ is closed. For fans closedness and transversal closedness coincide. Although the following theorem assures only the transversal closedness of a submultifan, the submultifan constructed in the proof is such that the proof extends the above quoted theorem of Nogura and Tanaka.

Theorem 3.1. *Each untraversable multifan in a regular Fréchet space includes a transversally closed submultifan.*

Proof. Let $f:T \to X$ be such a multifan. It is enough to show that there exists a neighborhood Q of $f(\emptyset)$ such that the restriction $f:f^{-}(Q) \to X$ has the property

$$t_k \sqsupseteq n_k, \quad \lim_k n_k = \infty \quad \text{and} \quad \lim_k f(t_k) = x \Longrightarrow x = f(\emptyset),$$

and then to induce on the rank of $t \in T$.

Suppose that on the contrary, for every closed neighborhood Q of $f(\emptyset)$, there exist $t_k^Q \sqsupseteq n_k^Q$ such that (n_k^Q) converges to ∞, $f(t_k^Q)$ converges to x_Q and $x_Q \neq f(\emptyset)$. Since $x_Q \in Q$, we have

$$f(\emptyset) \in \operatorname{cl}\{x_Q\colon \operatorname{cl} Q = Q \in \mathcal{N}(f(\emptyset))\}$$

and, by Fréchetness, there is a sequence $x_m = x_{Q_m}$ converging to $f(\emptyset)$. Therefore for each m, there exists $k(m)$ such that if $k \geqslant k(m)$, then $n_k^{Q_m} > m$. Let $t_{(m,k)} = t_k^{Q_m}$. By Fréchetness, there is a sequence $(t_{(m_p,k_p)})$ with $\lim_p m_p = \infty$ and $\lim_p f(t_{(m_p,k_p)}) = f(\emptyset)$. But

$$m_p < n_{k_p}^{Q_{m_p}} \quad \text{and} \quad n_{k_p}^{Q_{m_p}} \sqsubseteq t_{(m_p,k_p)}$$

which is impossible because f is untraversable. $\square$

By definition $(f \otimes g)(t) = (f(t), g(t))$.

Lemma 3.2. *Let X, Y be regular Fréchet spaces. If $f:T \to X$ is an untraversable multifan at x and $g:S \to Y$ is an untraversable arrow at y, then there exist a tree R, a submultifan $f_0:R \to X$ of an initial restriction of f and a subarrow $g_0:R \to Y$ of an initial restriction of g such that the diagonal multisequence $f_0 \otimes g_0$ fulfills*

$$\sigma(f_0 \otimes g_0) \geqslant 1 + (\lambda(f) \wedge \mu(g)). \tag{3.1}$$

Proof. We shall induce on $\alpha = \lambda(f) \wedge \mu(g)$. If $\alpha = -1$, then $\mu(g) = -1$, hence $r(g) = 0$; we put $g_0 = g$ and define f_0 as the initial restriction of rank 0 of f. Then $\sigma(f_0 \otimes g_0) \geqslant 0$.

If $\alpha = 0$, then $\mu(g) \geqslant 1$ and $\lambda(f) = 0$; take for g_0 the initial restriction of rank 1 of g and put $f_0 = f$.

Let $\alpha > 0$ and suppose that the property holds for each $\beta < \alpha$. By definition, there exist sequences (n_k) in T and (m_k) in S such that $\alpha_k = r(n_k; f) \wedge r(m_k; g)$ fulfills $\lim_k(\alpha_k + 1) = \alpha$. By inductive assumption, for each $k \in \mathbb{N}$, there exists a tree R_k, a subarrow $f_{0,k} : R_k \to X$ of an initial restriction of $f_k : T_k \to X$ (where $T_k = \{t\colon (n_k, t) \in T\}$ and $f_k(t) = f(n_k, t)$) and a submultifan $g_{0,k} : R_k \to Y$ of an initial restriction of $g_k : S_k \to Y$ (where $S_k = \{s\colon (m_k, s) \in S\}$ and $g_k(s) = g(m_k, s)$) such that

$$\sigma(f_{0,k} \otimes g_{0,k}) \geqslant 1 + \bigl(r(f_k) \wedge r(g_k)\bigr) = 1 + \alpha_k.$$

We define

$$R = \{\emptyset\} \cup \bigl\{(k, r)\colon\ r \in R_k,\ k \in \mathbb{N}\bigr\},$$

$h(k) = n_k$, $h(k, r) = h_k(r)$ (where $h_k : R_k \to T_k$ is such that $f_{0,k} = f_k \circ h_k$), $l(k) = m_k$, $l(k, r) = l_k(r)$ (where $l_k : R_k \to S_k$ is such that $g_{0,k} = g_k \circ l_k$).

By Theorem 3.1, there exists a submultifan $f_1 : P \to X$ of $f_0 : R \to X$ such that f_1 is transversally closed. Let $\widetilde{h} : P \to R$ be such that $f_1 = f \circ \widetilde{h}$ and put $g_1 = g_0 \circ \widetilde{h}$. Therefore

$$\sigma(f_1 \otimes g_1) = \lim_n \bigl(\sigma(f_{0,\widetilde{h}(n)} \otimes f_{0,\widetilde{h}(n)}) + 1\bigr), \tag{3.2}$$

hence by inductive assumption, $\sigma(f_1 \otimes g_1) \geqslant \lim_k(1 + \alpha_k + 1)$. Now if $0 < \alpha < \omega_0$, then $\alpha_k = \alpha - 1$ for almost k, if $\alpha = \omega_0$, then $\lim_k(1 + \alpha_k + 1) = \omega_0 = 1 + \omega_0$ and finally if $\alpha \geqslant \omega_0$, for almost all k, one has $1 + \alpha_k = \alpha_k$ and $\lim_k(\alpha_k + 1) = \alpha$. Therefore (3.2) holds. □

As a consequence of Lemma 3.2, we have

Theorem 3.3. *If X and Y are regular Fréchet spaces, then*

$$\sigma(x, y) \geqslant 1 + \bigl[\bigl(\lambda(x) \wedge \mu(y)\bigr) \vee \bigl(\mu(x) \wedge \lambda(y)\bigr)\bigr]. \tag{3.3}$$

4. Transverse and sequentially transverse topologies

An upper bound for the sequential order of products is given in Section 5 in the case of subtransverse topologies that we define below. A topology is *transverse* if for every injective sequence (x_n) converging to x, there exists a sequence Q_n with $Q_n \in \mathcal{N}(x_n)$ such that

$$\lim_n Q_n = x,$$

i.e., for each $Q \in \mathcal{N}(x)$ there exists $n_Q \in \mathbb{N}$ with $Q_n \subset Q$ for $n \geqslant n_Q$. A topology is *subtransverse* if for every injective sequence (x_n) converging to x, there exists a subsequence (n_k) and a sequence Q_k with $Q_k \in \mathcal{N}(x_{n_k})$ such that $\lim_k Q_k = x$. A convergent bisequence

$$x_{(n,k)} \xrightarrow[k]{} x_n \xrightarrow[n]{} x \tag{4.1}$$

with $\lim_n x_{(n,k_n)} = x$ for every sequence (k_n) is called *transverse*. One observes that the topology induced on such a bisequence is first-countable. A topology is *sequentially transverse* if for every convergent injective bisequence, there exists $f:\mathbb{N} \to \mathbb{N}$ such that the bisequence restricted to $x_{(n,k)}$ such that $k \geqslant f(n)$ for all n is transverse. A topology is *sequentially subtransverse* if every convergent bisequence admits a transverse subbisequence.

In [16] Popov and Rančin say that a topological space X is a *Φ-space* if for every $A \subset X$ and for each $x \in \operatorname{cl} A$, there exists a sequence (Q_n) of open sets such that $\lim_n Q_n = x$ and $Q_n \cap A \neq \emptyset$ for each n. In [4, Proposition 7] it is shown that a topological space is a Φ-space if and only if it is a sequential subtransverse space.

Each sequential sequentially subtransverse space is Fréchet. The Simon topology (Proposition 4.2) is an example of a Fréchet not sequentially subtransverse space. On the other hand,

Proposition 4.1. *Each Fréchet space with a point countable k-network is sequentially transverse.*

This fact follows from [13, Lemma 2.6] where Nogura and Shibakov prove (more than they announce) that in each Fréchet space with a point countable k-network for every convergent bisequence (4.1), there exists $h:\mathbb{N} \to \mathbb{N}$ such that

$$\{x\} \cup \{x_n\colon\ n \in \mathbb{N}\} \cup \{x_{(n,k)}\colon\ k \geqslant h(n),\ n \in \mathbb{N}\}$$

is compact and the points of the form $x_{(n,k)}$ are isolated in it.

Example 4.4 shows that there exist Fréchet transverse topologies without a point countable k-network. It is shown in a forthcoming paper [4, Example 9] that the Σ-product of uncountably many copies of the discrete two-point space is a Fréchet sequentially subtransverse not subtransverse space. Proposition 10 in [4] implies the existence of subtransverse not transverse spaces.

Recall that a closed continuous image of a metrizable space is called a *Lašnev space*. It is known that every Lašnev space is a Fréchet space with a point countable k-network [6]. In [16] Popov and Rančin show that each Lašnev space is subtransverse. Unaware of their result, we have proved (Proposition 4.3) that Lašnev spaces are transverse.

A compact Fréchet topology whose square is not Fréchet is called a Simon topology. The existence of Simon topologies is proved in [17].

Proposition 4.2. *Each Simon topology is not sequentially subtransverse.*

Proof. Let X, Y be Simon topological spaces. Since $X \times Y$ is not Fréchet, there exists in it an irreducible convergent bisequence (4.1). Of course, its every subbisequence is irreducible. Now for almost every n, either $x_n \neq x$ or $y_n \neq y$. Thus assuming that there are infinitely many n for which $x_n \neq x$, we extract the corresponding subbisequence. If X were sequentially subtransverse, then there would be a transverse convergent subbisequence of $x_{(n,k)} \to_k x_n \to_n x$; if the subbisequence of $y_{(n,k)} \to_k y_n \to_n y$ corresponding to the same indices is such that $y_n \neq y$ for infinitely many n, then we extract a transverse subbisequence. But this contradict the irreducibility of the original bisequence (4.1). If the above subbisequence of $y_{(n,k)} \to_k y_n \to_n y$ is a fan, then there is a sequence $(x_{(n_p,k_p)})_p$ converging to y with n_p tending to ∞, because a compact Fréchet topology is strongly Fréchet [1]. This contradicts the irreducibility of the original bisequence (4.1). □

Proposition 4.3. *Each Lašnev space is transverse.*

Proof. Let W be a metrizable space and $f:W \to X$ be a continuous closed map (onto X) and let (x_n) be an injective sequence converging to x in X. In other words f^- is an upper semicontinuous relation and the restriction of the topology of X to $\{x\} \cup \{x_n\colon n \in \mathbb{N}\}$ is first-countable. Hence by the active boundary theorem of Choquet [3, Théorème 3], there is a compact subset K of $f^-(x)$ such that for each $Q \in \mathcal{N}(K)$ there exists n_0 such that $f^-(x_n) \subset Q$ for each $n \geqslant n_0$. Because K is compact, there is a countable (decreasing) base (P_k) of $\mathcal{N}(K)$, hence for each k there is $n(k)$ so that for each $n \geqslant n(k)$, $f^-(x_n) \subset P_k$. Since f^- is upper semicontinuous at each x_n, for each $n \geqslant n(k)$, there is $Q_n^k \in \mathcal{N}(x_n)$ with $f^-(Q_n^k) \subset P_k$. By setting $Q_n = Q_n^k$ for $n(k) \leqslant n < n(k+1)$, we get that $f^-(Q_n) \subset P_k$ for $n \geqslant n(k)$ so that, by the continuity of f, the sequence (Q_n) converges to x, proving that the space is transverse. □

Example 4.4. *A transverse topology without point countable k-network.* Consider the *point cofinite topology* on an uncountable set X. Let x_0 be a distinguished point of X. The neighborhoods of x_0 are the sets containing x_0 and having finite complement. All the other points are isolated. All the free sequences converge to x_0, thus the topology is transverse. On the other hand, for each k-network on this space, x_0 belongs to uncountably many of its members.

5. Upper bounds for subtransverse topologies

Lemma 5.1. *Let X, Y be subtransverse spaces. If $f \otimes g : T \to X \times Y$ is an admissible monotone multisequence of sequential order α, then there exist a tree R of rank α and multisequences $f_0 : R \to X$ and $g_0 : R \to Y$ which, restricted to $R \setminus \max R$, are an untraversable multifan and an untraversable arrow, and such that $f_0 \otimes g_0$ is an irreducible transmultisequence of $f \otimes g$.*

Proof. Let $f \otimes g : T \to X \times Y$ be an admissible multisequence of sequential order α converging to (x, y). If $\alpha = 0$, set $f_0 = f$ and $g_0 = g$; then the set of nonmaximal indices is empty. If $\alpha = 1$, then for infinitely many n either $f(\emptyset) \neq f(n)$ or $g(\emptyset) \neq g(n)$. Let h be a sequence of natural numbers such that one of the above holds for each $n \in h(\mathbb{N})$, say $g(\emptyset) \neq g(n)$. Then $g_0 = g \circ h$ is an injective sequence (an untraversable arrow of rank 1) and the restriction of $f_0 = f \circ h$ to $\{\emptyset\}$ is a fan. If $\alpha > 1$, then for almost every n,

$$\big(f(n) = x \text{ and } g(n) \neq y\big) \quad \text{or} \quad \big(f(n) \neq x \text{ and } g(n) = y\big). \tag{5.1}$$

Indeed, suppose that this does not hold. If there are infinitely many n for which $f(n) = x$ and $g(n) = y$, then $f \otimes g$ is not injective. Suppose that there are infinitely many n for which $f(n) \neq x$ and $g(n) \neq y$. By the subtransversality of X and Y, there are $P_k \in \mathcal{N}(f(n_k))$ and $Q_k \in \mathcal{N}(g(n_k))$ such that

$$\lim_k P_k = x \quad \text{and} \quad \lim_k Q_k = y. \tag{5.2}$$

Since $\alpha > 1$, there exists a sequence $(t_k) \in \max T$ with $t_k \sqsupseteq n_k$ and $\lim_k n_k = \infty$ such that $f(t_k) \in P_k$ and $g(t_k) \in Q_k$ and $\sup_k(r(t_k) + 1) < \alpha$. Hence, by admissibility, $\lim_k(f(t_k), g(t_k))_k \neq (x, y)$, but by (5.2), $\lim_k f(t_k) = x$, $\lim_k g(t_k) = y$, a contradiction.

One proposition of the alternative (5.1) holds for infinitely many n, say, $x = f(n)$ and $y \neq g(n)$ and $r(n) < r(\emptyset)$. Let h be an injective sequence ranging over the set of all such n. We set $f_0(n) = f(h(n))$.

If $\alpha = 2$, then for each n, let $f_n(k) = f(h(n), k)$ and $g_n(k) = g(h(n), k)$. Almost all f_n are almost injective, for otherwise $f \otimes g$ would be of rank 1 (the sequences $g_n(k)$ need not be injective). By extending h to the effect that $h(m, k) = (h(m), k)$, we complete the proof for $\alpha = 2$.

By the inductive assumption, for each such an n, there exist a tree R_n of rank $r(n)$ and multisequences $f_n : R_n \to X$ and $g_n : R_n \to Y$, one an untraversable multifan, the other an untraversable arrow (for non maximal points of R_n) such that $f_n \otimes g_n$ is a transmultisequence of $f \otimes g$ restricted to $\{t \in T\colon\ t \sqsupseteq n\}$.

Let $\alpha > 2$ and suppose that the lemma holds for every $\beta < \alpha$. Consider the set B of those $t \in T$ for which $f(t) = f(\emptyset)$. Of course, $\emptyset \in B$ and $\mathbb{N} \subset B$. Since T is well-capped, there exist $t_k \in \max B \cap T$ with $t_k \sqsupseteq n_k$ with $\lim_k n_k = \infty$ and with $r(t_k)$ monotone. Clearly

$$\lim_n \bigl(r(t_n) + 1\bigr) = \alpha, \tag{5.3}$$

for otherwise the rank of $f \otimes g$ would be strictly less than α.

If $t_n \notin \max T$ (there are infinitely many such t_n), then by subtransversality there is a sequence $(k_p^n)_p$ such that $g(t_n, k_p^n) \neq g(t_n)$ for all p and such that $\lim_n g(t_n, k_p^n) = g(\emptyset)$. Let h be an injective bisequence such that $(h(m))_m$ ranges in the set of the considered t_n and let $h(m, p) = (t_n, k_p^n)$ if $t_n = h(m)$.

By the inductive assumption, we can extend h to a tree R of rank α so that the product of $f_0 = f \circ h$ and $g_0 = g \circ h$ be a monotone transmultisequence of $f \otimes g$ and that for each n, either $f_{0,n}(r) = f_0(n, r)$ is an arrow (for nonmaximal indices) or $g_{0,n}(r) = g_0(n, r)$ is a multifan (for nonmaximal indices).

We claim that for each n, the multisequence $f_{0,n}$ is an arrow and the multisequence $g_{0,n}$ is a multifan. Indeed, since for each k, one has $f_{0,n}(\emptyset) \neq f_{0,n}(k)$, by (5.1), $g_{0,n}(\emptyset) = g_{0,n}(k)$. It follows that f_0 is a multifan and g_0 is an arrow (for nonmaximal indices). They are untraversable. In fact, let r be a nonmaximal index; assume that it is of even length (the case of odd length is perfectly symmetric), then $f_{0,r}$ given by $f_{0,r}(s) = f_0(r, s)$

is a monotone (injective) multifan, and $g_{0,r}$ given by $g_{0,r}(s) = g_0(r,s)$ is a monotone (injective) arrow. As $g_0(r) \neq g_0(r,n)$, one has

$$r_n \sqsupseteq (r,n) \Longrightarrow \lim_n g_0(r_n) = g_0(r). \tag{5.4}$$

For each n, one can find a sequence $(r_{(n,k)})_k$ in $\max R$ with $(f_0(r_{(n,k)}))_m$ converging to $f_0(r_n)$. If a subsequence of $f_0(r_n)$ converges to $f_0(n)$, then by the (sequential) subtransversity of X, there would be a transverse subbisequence of $f_0(r_{(n,k)}) \to_k f_0(r_n) \to_n f_0(r)$. By (5.4), the index r would be of sequential order 1 with respect to $f_0 \otimes g_0$. As r is arbitrary nonmaximal, this is contrary to the assumption that $\alpha > 2$. In conclusion, since $f_{0,n}$ is an untraversable arrow and $g_{0,n}$ is an untraversable multifan, $f_0 \otimes g_0$ is irreducible. □

Let us comment on the above proof.

Proposition 5.2. *If $f \otimes g : T \to X \times Y$ is admissible monotone and if $R \subset T$ is a tree such that for each $r \in R$, $f(r) = f(\emptyset)$, then the set*

$$R \setminus \big(\max R \setminus \max(R \setminus \max R)\big) \tag{5.5}$$

consists of elements of limit rank with respect to T.

Proof. Indeed, it is enough to show that (5.5) holds for multisequences indexed by R. So let $\alpha = r(\emptyset; T)$ be nonlimit and let $x = f(\emptyset) = f(n) = f(n,k)$ for each n and k. Because of the admissibility of the multisequence $(f(t), g(t))_{t\in T}$, if $(f(n,k_n), g(n,k_n))$ converges to $(f(\emptyset), g(\emptyset))$ for some sequence k_n, then $\lim_n (r((n,k_n); T) + 1) = \alpha$.

On the other hand, $r((n,k_n); T) < r(n; T) = \alpha - 1$; if not, then $r((n,k_n); T) + 1 = \alpha - 1$, a contradiction. Accordingly, $g(n,k_n)$ never converges to $g(\emptyset)$. But as Y is a Fréchet space and

$$g(\emptyset) \in \operatorname{cl}\{g(n,k):\ n,k \in \mathbb{N}\},$$

there is a subsequence $g(n_p, k_p)$ (with (n_p) tending to infinity) converging to $g(\emptyset)$, a contradiction. □

In general it is not possible to replace, in Lemma 5.1, a transmultisequence by a submultisequence.

Example 5.3. Let T be a tree such that $r(n;T) = \omega_0 n$ for each $n \in T$. Let $f : T \to X$ be such that $f(\emptyset) = f(t)$ if $t \sqsupseteq n$ and $r(t;T) > \omega_0(n-1)$ and the remaining indices form an irreducible multifan; let $g : T \to X$ be a multisequence such that $\lim_k g(t_k) = g(t)$ if $t_k \sqsupseteq (t, n_k)$ and $r(t_k; T) > \omega_0(n-1)$ and the remaining indices form an irreducible arrow. Of course, the multisequences f and g do not admit submultisequences f_0 and g_0 that are a multifan and an arrow, respectively.

Theorem 5.4. *If X and Y are sequential subtransverse spaces, then*

$$\sigma(x,y) \leqslant 1 + \big[\big(\lambda(x) \wedge \mu(y)\big) \vee \big(\mu(x) \wedge \lambda(y)\big)\big]. \tag{5.6}$$

Proof. Let $\alpha < \sigma(x,y)$. Then by definition, there exists a subset A of $X \times Y$ such that $\sigma((x,y);A) = \alpha$ and by Proposition 1.3, there exists a monotone admissible multisequence $f \otimes g : T \to X \times Y$ of rank α. If $\alpha = 1$, then at least one of the sequences f and g admits an injective subsequence and thus (5.6) holds. If α is finite, then by Lemma 5.1, the multisequence $f \otimes g$ admits a transmultisequence $f_0 \otimes g_0$ of rank α (actually a submultisequence since the rank is finite) such that one of f_0, g_0 is an untraversable multifan (say, f_0) and another is an untraversable arrow. If α is even, then because of (2.1), $r(f_0) = \lambda(f_0)$ and $r(g_0) = \mu(g_0) + 1$ and if it is odd, then $r(f_0) = \lambda(f_0) + 1$ and $r(g_0) = \mu(g_0)$. Therefore (5.6) holds. If α is infinite, then by (2.1), the rank coincides with the fascicularity and sagittality respectively of f_0 and g_0 that have been constructed by virtue of Lemma 5.1. □

It follows from Lemma 5.1 that each monotone admissible multisequence in a product of two transverse spaces admits an irreducible transmultisequence of the same sequential order.

If in Lemma 5.1 we consider in particular a monotone irreducible multisequence $f \otimes g$, then we can weaken the assumption on X and Y and strengthen the conclusion that $f_0 \otimes g_0$ a submultisequence (rather than a transmultisequence) of $f \otimes g$:

Theorem 5.5. *Let X, Y be sequential sequentially subtransverse and let $f \otimes g : T \to X \times Y$ be a monotone irreducible multisequence of rank $\alpha > 1$. Then there exist a tree R and multisequences of rank α, $f_0 : R \to X$ and $g_0 : R \to Y$ for which the restriction to $R \setminus \max R$ of one of them is an untraversable multifan and of the other an untraversable arrow such that $f_0 \otimes g_0$ is a submultisequence of $f \otimes g$.*

Proof. Denote $x = f(\emptyset)$, $y = g(\emptyset)$. For $\alpha = 1$, the property holds without any assumption on X and Y. If $\alpha \geqslant 2$, then (5.1) holds for almost all n. In fact $f(n) = x$ and $g(n) = y$ cannot happen for infinitely many n because of irreducibility: suppose that $f(n) \neq x$ and $g(n) \neq y$ for infinitely many n (say for an injective sequence (n_p)). Then by sequential subtransversality, there is an injective mapping $h : \mathbb{N} \cup \mathbb{N}^2 \to \mathbb{N} \cup \mathbb{N}^2 \subset T$ such that $h(p) = n_p$, $h(p,q) = (n_p, k_q)$ with $\lim_q k_q = \infty$, $\lim_p f(h(p,q_p)) = x$ and $\lim_p g(h(p,q_p)) = y$, contrary to the irreducibility of $f \otimes g$.

Consequently (5.1) holds for almost every n. Therefore there are submultisequences f_0 and g_0 of f and g, respectively, such that one of the alternatives of (5.1) holds for each n, say, $f_0(n) = x$ and $g_0(n) \neq y$. Now for almost each n, there exists $k(n)$ such that $f_0(n) \neq f_0(n,k)$ for almost each $k \geqslant k(n)$, because otherwise by subtransversality of Y, we should find sequences (n_p), (k_p) such that $\lim_p g_0(n_p, k_p) = y$, contrary to the irreducibility of $f_0 \otimes g_0$.

Let $\alpha > 2$ and suppose that the property holds for each $\beta < \alpha$. Then for almost every n, there exists $k(n)$ such that $g_0(n) = g_0(n,k)$ for each $k \geqslant k(n)$, because of the property (5.1) (applied for each fixed n and with k replacing n in the formula) and the fact that $f_0(n) \neq f_0(n,k)$ for such n and k. Now we use the inductive assumption to complete the construction of f_0 and g_0. □

6. Precise bounds for Lašnev topologies

As already mentioned, Lašnev spaces are normal Fréchet and transverse. Hence by Theorems 3.3 and 5.4, we have

Theorem 6.1. *If X and Y are Lašnev spaces, then*

$$\sigma(x,y) = 1 + \big[\big(\lambda(x) \wedge \mu(y)\big) \vee \big(\mu(x) \wedge \lambda(y)\big)\big]. \tag{6.1}$$

On the other hand,

Theorem 6.2. *For every ordinal $\alpha \leqslant \omega_1$, there exists a Lašnev space such that the sequential order of its square is α.*

Proof. Consider the following map $f : \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \to [0,1[$:

$$f(\emptyset) = 0, \quad f(n) = \frac{1}{2^{n+2}}, \quad f(n,t) = f(n)\big(1 + f(t)\big).$$

Let T be a tree of rank $\alpha < \omega_1$. If α is even (respectively odd), then let X be the image by f of the elements of T of even (respectively odd) length and $W = f(T) \setminus X$. Equip W with the topology induced from the unit interval. Consider on X the quotient topology determined by the map $g : W \to X$ given by

$$g\big(f(t,n)\big) = f(t).$$

Then X becomes an (untraversable) multifan of rank α converging to 0. We show that g is a closed map (hence X is a Lašnev space).

Let $O \in \mathcal{N}(g^-(f(t)))$. Then for each n, the set O is a neighborhood of $f(t,n)$. Now the set $\{f(t,n)\colon n \in \mathbb{N}\}$ is a discrete subset of (the metric space) W. Let $k(n)$ be the least integer such that the interval

$$I_n = \big[f(t,n), f\big(t,n,k(n)\big)\big[\subset O.$$

Then $\{I_n\}$ is a discrete family of closed mutually disjoint neighborhoods respectively of $f(t,n)$ and $g^-(g(I_n)) = I_n$ for each n. Consequently, $I = \bigcup_n I_n$ is a closed neighborhood of $g^-(f(t))$ and $g(I)$ is a closed neighborhood of $f(t)$ for which $g^-(f(t)) \subset g^-(g(I)) \subset O$, hence g is a closed map.

We have constructed a Lašnev space in which $\lambda(0) \wedge \mu(0) = \alpha$ and for all other points x, one has $\lambda(x) < \alpha$ and $\mu(x) < \alpha$.

In case where $\alpha = \omega_1$, substitute in the proof T by the whole set of finite sequences on $\mathbb{N}$.

By Theorems 3.3 and 5.4 the proof is complete. □

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